though the play of forces was very similar to that in Mr. Hinde's experiment, energy was fed in from outside by the agent maintaining the azimuthal oscillations.

Another example of progressive motion obtained by oscillatory motion of the axis of the spin of a gyrostat is Dr. Gray's "walking gyrostat."

I have described some of these devices in a work on 'Gyrostatics,' now in the press.

The Behaviour of Scattering Media in Fully Diffused Light.

By H. J. Channon, F. F. Renwick, and B. V. Storr.

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Introduction.

The present paper is the outcome of a discussion which arose between the authors after the publication of a paper by O. Bloch and F. F. Renwick,* which dealt with the subject of the opacity of diffusing media, chiefly from the practical standpoint, and included also lengthy extracts from the Reports of a Committee of the American Society of Illuminating Engineers.†

On finding that we were unable to reconcile the experimental data with the theory included in the above paper, we were gradually led into the attempt to find solutions of some of the problems involved in this complex subject, starting only with the indisputable fact that when light falls upon a finite thickness of a scattering medium, part is rejected, part extinguished, and part transmitted. In the present communication we have confined ourselves to a study of the action of such media upon light which is assumed to be already completely diffused.

Definitions.

Let MN represent a large sheet of a diffusing medium of uniform thickness and quality, upon which completely diffused light of intensity I_0 falls.

Let the intensity of the light returned in directions opposed to I_0 be I_r, and the intensity of the light which escapes after passage through the sheet be I_t. The fraction I_r/I_o of the incident light may be called the "rejecting power" or the "rejectance" of the sheet, and will be denoted by R.

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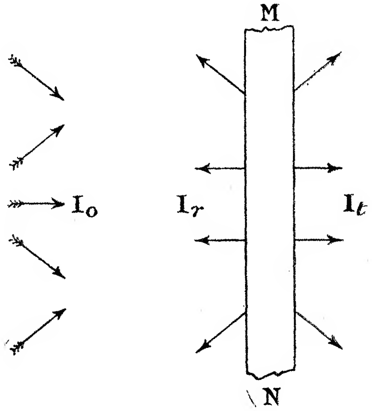
* 'Photo. Journ.,' vol. 56, pp. 49–65.

† 'Trans. Amer. Soc. Ill. Eng.,' vol. 10, No. 5, pp. 353–402 (1915).



The ratio of the light issuing from, to that impinging on the sheet, viz., I_t/I_o, we will call the "transmittance," while the inverse of this, I_o/I_t, we will call the "obstructance" and denote by Ω.

We then have by definition the following relationships:—

$$I_r/I_o = R \text{ (rejectance)},$$

$$I_o/I_t = \Omega \text{ (obstructance)}.$$

It is convenient also to express the ratio between the unrejected portion of the incident light and the light issuing by the symbol "C" and the term "contendance."

It should be noted that the rejectance (R) includes all light returned in directions opposed to the incident (diffused) light, no matter by what circuitous path within the material it may have travelled. It seems therefore undesirable to employ the term "reflection" in describing it.* It should also be observed that a medium which neither rejects nor absorbs any of the incident light would have an "obstructance" and a "contendance" both = 1.

The equation $C = I_o(1-R)/I_t$ forms the starting point of our theoretical discussion; in order to eliminate everything not a property of the material, we will write this in the form

$$\Omega = C/(1-R). \tag{1}$$

$$\text{Obstructance} = \frac{\text{contendance}}{\text{selectance}}.$$

It is our object to show the changes that take place in the values of these properties with alteration of thickness and to reduce the relationships to their most general forms. Since, by definition, the above equation (1) is independent of the thickness of the sheet under consideration, we can write

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* *Vide* Rayleigh, 'Phil. Mag.,' vol. 41, p. 107 (1871).

$$\Omega_t = C_t/(1-R_t), \text{ whatever the value of } t.$$

Now suppose two sheets of any diffusing media m and n to be placed together to form a single sheet. Let Ω_m and R_m be the obstructance and rejectance respectively of the first sheet m, and Ω_n and R_n those of the second sheet n through which the light issues.

Then if light of intensity I be transmitted by m and fall upon n, we have an infinite geometrical series of "inter-rejections" which sum up, in the direction of m, to

$$I\,R_n/(1-R_mR_n).$$

In passing out through m this will be reduced to

$$I\,R_n/(1-R_mR_n)\,\Omega_m,$$

which will be the increment in the rejected light due to the backing up of m by n. But $I = I_o/\Omega_m$, where I_o = the light intensity incident upon m. Hence the total rejectance of the two sheets together will be

$$R_m+R_n/\Omega_m^2(1-R_mR_n).$$

Hence, generally, the rejectance from m of a composite layer of two diffusing media m and n in contact, whether of similar or dissimilar characters,

$$= R_{m+n} = R_m+R_n/\Omega_m^2(1-R_mR_n). \qquad (2)$$

Similarly the obstructance of the two sheets $m+n$ together

$$\Omega_{m+n} = (1-R_mR_n)\,\Omega_m\Omega_n. \qquad (3)$$

The second member of this equation is symmetrical with respect to m and n and therefore the obstructance of a composite sheet consisting of two diffusing media is independent of their arrangement.

If the sheets m and n consist of the same material, R_{m+n} will equal R_{n+m} and from (2)

$$R_{m+n}(1-R_mR_n) = R_m+R_n\,\{1/\Omega_m^2-R_m^2\},$$

the first member of which in these circumstances is also symmetrical with respect to m and n. We may, therefore, interchange m and n throughout and equate the results, thus,

$$R_m+R_n\,(1/\Omega_m^2-R_m^2) = R_n+R_m\,(1/\Omega_n^2-R_n^2),$$

or

$$\{1+R_m^2-1/\Omega_m^2\}/R_m = \{1+R_n^2-1/\Omega_n^2\}/R_n,$$

which is, therefore, a constant, independent of the thickness, for any diffusing medium.

Note.—Up to this point we have employed the same mode of reasoning as Stokes followed in his paper "On the Intensity of the Light Reflected from or Transmitted through a Pile of Plates."*

Writing this expression in the form $(1+R_t^2-1/\Omega_t^2)/R_t$ and inserting the values corresponding to $t = \infty$, it follows that the value of Stokes' constant is $(R_\infty + 1/R_\infty)$, where R_∞ is the rejectance of an infinitely thick layer.

Suppose a sheet of a uniformly lighted diffusing medium of rejectance ρ to be used as source of light, transmitting completely diffused light of intensity I'.

Let this light at once encounter another sheet of rejectance R and obstructance Ω; then the light intensity transmitted by the combination $I_1 = I'/(1-\rho R)\Omega$.

The ratio $I'/I_1 = (1-\rho R)\Omega$ may be called the "relative obstructance" O of the added sheet (*i.e.*, the obstructance of this sheet relative to ρ).

The inverse of this may be written T and called the relative "transmittance" while the logarithm of O to the base 10 may be denoted by D and called the relative obstructance logarithm of the sheet to diffused light.

Measurements made under these conditions have for many years been denoted by the above symbols and it seems, therefore, desirable to retain them but to abandon the incorrect terms, opacity for O, transparency for T, and density for D, usually given to them, in favour of those suggested.

In the case of measurements of black scattering media like photographic negatives, the value of R is usually so small that it may be permissible in ordinary practice to assume that it is zero and to regard the measured values as representing the true constants of the silver deposits, ignoring the complication due to the properties of the diffusing medium in the photometer.

We have seen that, generally, the relative obstructance of a sheet of diffusing medium m,

$$= O_m = (1-\rho R_m)\,\Omega_m, \tag{4}$$

where ρ is the rejectance of the photometer diffusing medium p.

The combined rejectance, on the side m, of m and p together will be (from equation 2)

$$R_{m+p} = R_m + \rho/(1-\rho R_m)\,\Omega_m^2.$$

Now if I_1 be the light intensity issuing through the first specimen m, that leaving a second superposed specimen n will be, by the usual process of reasoning,

$$I_2 = I_1/(1-R_n R_{m+p})\,\Omega_n.$$

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* 'Roy. Soc. Proc.,' vol. 11, p. 545 (1860–62).

But for the first specimen m

$$I_1 = I'/(1-\rho R_m)\,\Omega_m,$$

where I′ is the light intensity emitted by the bare photometer diffuser.

Hence $$O_{m+n} = \Omega_n O_m (1-R_m R_n) - \rho R_n \Omega_n/\Omega_m. \qquad (5)$$

In the special case in which $R_n = R_m = R_1$ and $O_m = O_n = O_1$ we have

$$O_2 = O_1\Omega_1(1-R_1^2)-\rho R_1,$$

and if also $R_1 = \rho$ then

$$O_2\,(\text{rel. to } \rho) = (1-\rho^2)^2\Omega_1^2-\rho^2.$$

But since, in this case, $O_1 = (1-\rho^2)\,\Omega_1$, it follows that, relative to ρ,

$$O_1^2 - O_2 = \rho^2. \qquad (6)$$

Hence if of three similar specimens we measure the relative obstructances of one specimen and of two specimens together, using completely diffused light emitted by the third specimen, we are in a position to derive at once the rejectance ρ of one specimen and therefrom, by means of the equations $O_1 = (1+\rho)C_1$ and $O_1 = (1-\rho^2)\Omega_1$ to deduce C_1 and Ω_1 also.

This result is obviously one of considerable value in practical photometry.

Some General Relationships Existing Between Values of Ω, etc.

1. Proof that

$$\Omega_{n+2}+\Omega_n = \Omega_{n+1}\times\text{constant},$$

where n, $n+1$, etc., refer to the number of similar layers of which a sheet of medium may be imagined to consist.

From equation 3 (p. 224),

$$\Omega_{n+2} = (1-R_1R_{n+1})\,\Omega_1\Omega_{n+1}, \qquad (a)$$

$$\Omega_{n+1} = (1-R_1R_n)\,\Omega_1\Omega_n. \qquad (b)$$

From equation 2 (p. 224),

$$R_{n+1} = R_1 + R_n/\Omega_1^2(1-R_1R_n),$$

(the layers being all alike, R_{n+1} will be the same as R_{1+n}).

Hence, in (a)

$$(1-R_1R_{n+1}) = 1-R_1^2-R_1R_n/\Omega_1^2(1-R_1R_n). \qquad (c)$$

From (b)

$$R_1R_n/(1-R_1R_n) = (\Omega_1\Omega_n-\Omega_{n+1})/\Omega_{n+1}. \qquad (d)$$

Therefore, from (c) and (d),

$$(1-R_1R_{n+1}) = \Omega_{n+2}/\Omega_1\Omega_{n+1} = 1-R_1^2-(\Omega_1\Omega_n-\Omega_{n+1})/\Omega_1^2\Omega_{n+1}.$$

Hence $$(1-R_1^2)\,\Omega_1^2\Omega_{n+1}-\Omega_1\Omega_n+\Omega_{n+1} = \Omega_1\Omega_{n+2},$$

or $$\Omega_{n+2}+\Omega_n = \{(1-R_1^2)\,\Omega_1+1/\Omega_1\}\,\Omega_{n+1}. \qquad (7)$$

Since R_1 and Ω_1 are the rejectance and obstructance of a single layer, the expression $[(1-R_1^2)\Omega_1+1/\Omega_1]$ is a constant. Consequently, we may write

$$\Omega_{n+2}+\Omega_n = K\Omega_{n+1}.$$

From this, by substitution, it is easy to prove also that

$$C_{n+2}+C_n = KC_{n+1} \quad \text{and} \quad O_{n+2}+O_n = KO_{n+1},$$

where K has the value already found above. Writing these in the form

$$(\Omega_{n+1}-\Omega_n)-(\Omega_n-\Omega_{n-1}) = (K-2)\Omega_n,$$

it is evident that the values of Ω, O, and C for 0, 1, 2, 3, etc., units of thickness form a series 1, Ω_1, $K\Omega_1-1$, $K^2\Omega_1-K-\Omega_1$, etc., whose differences of the second order constitute a new series identical with the original series multiplied by the factor $K-2$.

This property of a finite series is consistent only with the conclusion that the relationships between Ω, C, and O, and the variable t, are either simple exponential relationships or the sums or differences of two exponential expressions having the same indices, one positive, the other negative.

It is evident that the above series for 0, 1, 2, 3, etc., thicknesses is not a simple exponential series, and consequently the formulæ required must all be of the general form

$$\Omega_t\,(\text{or } C_t \text{ or } O_t) = Pe^{\lambda t}+Qe^{-\lambda t}, \tag{8}$$

which is the integral of $d^2\Omega/dt^2 = K\Omega$, P, Q and λ being constants.

Since all three series are identical except for the interchange of Ω_1, C_1, and O_1 it is clear that the evaluation of the constants for Ω will give those of the other two by simple substitution of C_1 and O_1 for Ω_1.

Confirmation of the above general equation is to be found in a paper by Prof. A. Schuster.* His solution for the emergent radiation R through any thickness t of a self-radiant foggy atmosphere is as follows:—

$$R = 2\alpha\frac{[(1+\alpha)e^{\alpha(k+s)t}+(1-\alpha)e^{-\alpha(k+s)t}]E+2(S-E)}{(1+\alpha)^2e^{\alpha(k+s)t}-(1-\alpha)^2e^{-\alpha(k+s)t}},$$

where E = total energy of radiation sent out by unit surface of a completely black surface,

S = total energy of radiation incident on unit surface of plane layer of foggy gas,

k = coefficient of absorption, s = coefficient of scattering,

$\alpha = \sqrt{[k/(k+s)]}$, t = thickness.

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* "Radiation through a Foggy Atmosphere," 'Astrophys. Journ.,' vol. 21, p. 1 (1905).

In our case E may be taken as zero, for we are not considering self-luminous or fluorescent media; hence Schuster's equation simplifies to

$$S/R = \Omega_t = [(1+\alpha)^2 e^{\alpha(k+s)t} - (1-\alpha)^2 e^{-\alpha(k+s)t}]/4\alpha,$$

and by writing

$$(1+\alpha)^2/4\alpha = P, \qquad -(1-\alpha)^2/4\alpha = Q,$$

and

$$\alpha(k+s) = \lambda,$$

Schuster's equation is the same as our general expression.

We are now in a position to evaluate the constants in the above general equation from the knowledge we possess of the values of the first few terms for $t = 0$, 1 and 2, namely, 1, Ω_1, and $K\Omega_1 - 1$.

Putting $t = 0$,

$$P+Q = 1; \qquad Q = 1-P.$$

Putting $t = 1$,

$$Pe^{\lambda} + (1-P)e^{-\lambda} = \Omega_1 \qquad \text{(or } C_1 \text{ or } O_1 \text{ resp.)},$$

$$P = (\Omega_1 - e^{-\lambda})/2\sinh\lambda; \qquad Q = (e^{\lambda} - \Omega_1)/2\sinh\lambda.$$

Putting $t = 2$,

$$(\Omega_1 - e^{-\lambda})e^{2\lambda}/2\sinh\lambda + (e^{\lambda} - \Omega_1)e^{-2\lambda}/2\sinh\lambda = K\Omega_1 - 1,$$

$$= 2\Omega_1\cosh\lambda - 1,$$

therefore

$$K = 2\cosh\lambda.$$

Hence the full expressions for Ω_t, C_t, and O_t are as follows:—

$$\Omega_t = (\Omega_1 - e^{-\lambda})e^{\lambda t}/2\sinh\lambda + (e^{\lambda} - \Omega_1)e^{-\lambda t}/2\sinh\lambda, \qquad (9)$$

$$C_t = (C_1 - e^{-\lambda})e^{\lambda t}/2\sinh\lambda + (e^{\lambda} - C_1)e^{-\lambda t}/2\sinh\lambda, \qquad (10)$$

$$O_t = (O_1 - e^{-\lambda})e^{\lambda t}/2\sinh\lambda + (e^{\lambda} - O_1)e^{-\lambda t}/2\sinh\lambda. \qquad (11)$$

$$\Omega_1 = C_1/(1-R_1), \qquad O_1 = (1-\rho R_1)\Omega_1 = (1-\rho R_1)C_1/(1-R_1),$$

where ρ is the rejectance of the photometer diffuser, Ω_1, C_1, and R_1 being the obstructance, contendance, and rejectance respectively of unit thickness of the medium, t being expressed in the same units.

Any of the above may be expressed in a number of different forms, of which the following should be mentioned:—

$$\Omega_t = [\Omega_1\sinh\lambda t - \sinh\lambda(t-1)]/\sinh\lambda, \qquad (12)$$

(with similar forms for C_t and O_t) as being the simplest for calculating numerical values.

The Change of R with t.

Since

$$\Omega_t = C_t/(1-R_t), \qquad R_t = 1 - C_t/\Omega_t.$$

Hence
$$R_t = 1 - \frac{(C_1 - e^{-\lambda})\,e^{\lambda t} + (e^{\lambda} - C_1)\,e^{-\lambda t}}{(\Omega_1 - e^{-\lambda})\,e^{\lambda t} + (e^{\lambda} - \Omega_1)\,e^{-\lambda t}}$$
$$= \frac{(\Omega_1 - C_1)\{e^{\lambda t} - e^{-\lambda t}\}}{(\Omega_1 - e^{-\lambda})\,e^{\lambda t} + (e^{\lambda} - \Omega_1)\,e^{-\lambda t}}$$
$$= \frac{2\,\Omega_1 R_1 \sinh \lambda t.}{(\Omega_1 - e^{-\lambda})\,e^{\lambda t} + (e^{\lambda} - \Omega_1)\,e^{-\lambda t}} \tag{13}$$
$$= \frac{O_1 R_1 \sinh \lambda t}{O_1 \sinh \lambda t - (1 - \rho R_1) \sinh \lambda\,(t-1)}. \tag{14}$$

If we rewrite equation (13) thus
$$R_t = R_1 \Omega_1 \frac{1 - e^{-2\lambda t}}{(\Omega_1 - e^{-\lambda}) + (e^{\lambda} - \Omega_1)\,e^{-2\lambda t}}$$
it becomes evident that when t is infinitely great
$$R_t = \Omega_1 R_1/(\Omega_1 - e^{-\lambda}) = R_\infty.$$
the maximum possible rejectance of the material with increasing thickness.

This constant enables us to derive a number of simpler expressions for R_t, and may itself be expressed in many different forms, of which
$$R_\infty = O_1 R_1/[O_1 - (1 - \rho R_1)\,e^{-\lambda}] \tag{15}$$
is useful for its practical application, while
$$R_t = 2 \sinh \lambda t/(e^{\lambda t}/R_\infty - R_\infty\,e^{-\lambda t}),$$
$$\text{or, if} \quad R_\infty = e^{\beta}, \qquad R_t = \sinh \lambda t/\sinh(\lambda t - \beta) \tag{16}$$
is a convenient form for calculating a number of values of R_t.

Lastly, it appears desirable to give the relations existing between Schuster's constants α, k, and s, and those we employ. They are expressed as follows:—
$$\Omega_1 = (1+\alpha)^2\,e^{\alpha(k+s)}/4\alpha - (1-\alpha)^2\,e^{-\alpha(k+s)}/4\alpha,$$
$$C_1 = \tfrac{1}{2}(1+\alpha)\,e^{\alpha(k+s)} + \tfrac{1}{2}(1-\alpha)\,e^{-\alpha(k+s)},$$
$$R_1 = \frac{(1-\alpha^2)\,2 \sinh \alpha\,(k+s)}{(1+\alpha)^2\,e^{\alpha(k+s)} - (1-\alpha)^2\,e^{-\alpha(k+s)}},$$
$$R_\infty = (1-\alpha)/(1+\alpha). \tag{17}$$

From equation (17) it is easy to calculate numerical values for α, and therefrom to derive k and s by means of $\lambda = k/\alpha$ and $\lambda = \alpha\,(k+s)$ when R_∞ is known.

Considerations affecting the Practical Problem of Measurement.

In order to determine the actual values of those properties of scattering media which have been considered in the foregoing theoretical discussion, it is necessary to employ a sufficiently large source of completely diffused light and specimens of the materials to be studied of sufficient area to avoid the serious errors which may arise with small specimens from loss of light at the edges.

The only convenient means available for getting uniform diffused light of constant intensity is to employ a good diffusing medium illuminated by a constant light source, and hence it is necessary in practice to measure values of O_t or D_t relative to a diffusing medium of known rejectance ρ. A photometer, therefore, to be suitable for this class of work should be provided with at least three exactly similar good diffusing screens, one of them to serve as the permanent diffusing screen of the instrument, and two others to be measured as specimens for determining the value of ρ in accordance with the equation $O_1^2 - O_2 = \rho^2$, already derived.

When other specimens of unknown properties are now measured in contact with the same diffusing screen, it is possible, from the results obtained and the value of ρ for the photometer, to deduce the desired values for any new material.

For this method it is necessary, as a mininum, to obtain by measurement the values of the relative obstructances for specimens both one unit and two units thick. The rejectance R_1 of unit thickness of the new material can then be derived from equation (5) in the form:—

$$R_1^2(O_1^2 - \rho^2) - R_1(O_2 - 1)\rho - (O_1^2 - O_2) = 0. \tag{18}$$

The values of C_1 and Ω_1 then follow from

$$O_1 = (1 - \rho R_1)\,C_1/(1 - R_1) = (1 - \rho R_1)\,\Omega_1,$$

while

$$K = 2\cosh\lambda = (O_2 + 1)/O_1.$$

From these data any desired values can be calculated with the aid of the equations given in the first part of this paper.

It is necessary to utter a warning concerning the extreme care necessary for determining the values of ρ and R_1 in the above manner. The results depend on the comparatively small differences between two much greater values, O_1^2 and O_2; consequently any errors in the values of O_1 and O_2 will be greatly increased in those of ρ and R_1 deduced from them.

If two photometer diffusing screens of widely different and accurately known rejectances ρ and ρ' are available, a simpler solution of the problem offers itself, having, moreover, the advantage of requiring only one specimen of the material. In these circumstances, if O and O' are the obstructances of the specimen relative to ρ and ρ' respectively, then the rejectance R of the specimen is given by

$$R = (O - O')/(\rho' O - \rho O'). \tag{19}$$

The method of measurement between two opal glasses, prescribed by Nutting,* which is supposed to give the absolute or "total" transparency to diffused light, can be shown in accordance with our theoretical discussion of the subject to give in reality the following highly complex function of the properties of both diffusers and specimen :—

$$\frac{\Omega_{m+p}}{\Omega_{m+n+p}} = \frac{\Omega_n (1-\mathrm{R}_m\mathrm{R}_p)}{\Omega_n^2 (1-\mathrm{R}_m\mathrm{R}_n)(1-\mathrm{R}_n\mathrm{R}_p)(1-\mathrm{R}_m\mathrm{R}_p)-\mathrm{R}_m\mathrm{R}_p},$$

where n refers to the specimen between the two diffusing plates m and p.

Such a result evidently bears no simple relation to the obstructance of the specimen, nor could its value be deduced even if R_m and R_p were known.

Experimental Confirmation of the Theory.

The paper already mentioned, published by one of us† in collaboration with O. Bloch, contains measurements of the relative obstructance logarithms of specimens of opal glass and various other granular media (suspensions in gelatine). These results led to the discovery of an empirical rule connecting relative obstructance with thickness (or concentration) of the medium, viz. :—

$$\mathrm{D} = at^b,$$

where a and b are constant for the medium and the given conditions of measurement. This formula was found to represent very closely the observed relationship, the agreement falling well within the limits of experimental error in most cases.

No physical explanation of this rule was apparent, so that it was deemed advisable to make further measurements to see if any departure from it could be established by exercising every care, and especially to test practically our theoretical deductions. A sheet of thin ordinary opal glass of good uniform quality was cut up into six circles, one being of 5 cm. diameter, the rest of 3½ cm. diameter. All of them were ground to a fine matt surface on both sides and brought to approximately equal and uniform thickness, except one (f) which was made specially thin. Measurements with a micrometer gauge showed that the extreme variation in thickness of any one specimen did not exceed 0·01 mm. Before measurement, all were thoroughly cleaned by boiling in strong mixed chromic and hydrochloric acids.

The following are the thicknesses of the specimens as the mean of about 10 measurements of each :—5 cm. circle, used as diffuser in the photometer, 0·969 mm. Specimen a, 0·962 mm.; b, 0·987 mm.; c, 0·989 mm.

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* 'Amer. Ill. Eng. Soc. Trans.,' vol. 10, p. 366 *et seq.*

† 'Photo. Journ.,' vol. 56, pp. 49–65.

d, 0·986 mm.; *e*, 0·952 mm.; *f*, 0·655 mm. The photometer used was of a simple type, operating on the principle of the law of inverse squares of distances of the illuminant. The illuminant was a fixed incandescent electric bulb having a compact gridiron filament, set at the back of the photometer box, which illuminated a piece of opal filling one-half of the field of a Lummer-Brodhun cube, and also emitted a beam along the photometer bench. A travelling carriage on the bench carried two mirrors, set at 90° to one another, by which the light was returned on a parallel path and illuminated the 5 cm. diameter opal disc already referred to, which was set in the hinged side of the photometer box close to the Lummer-Brodhun cube. The hinged side of the photometer box was double-walled, and so constructed that the 3½ cm. specimens placed between the double walls were in intimate contact with the 5 cm. disc of opal, but light from the central spot alone, about ¼ inch diameter, entered the photometer box, and was viewed in the Lummer-Brodhun cube.

The scale of the photometer bench, which was calibrated to read direct in relative obstructance logarithms, was carefully checked, and its setting relative to the diffusing opal was found correct by repeated measurements of the same specimen with several zero readings at widely different parts of the scale as well as by direct length measurements. No appreciable difference in the readings could be observed whether the surrounding, ¾ cm. wide, annulus of the instrument opal was masked or not. (Bloch and Renwick found that this makes a difference if the specimens are too small.)

A number of sets of observations upon these specimens were made both with simple dry contact and also with the piles of specimens and the diffuser in optical contact throughout by means of films of cedar-wood oil. Very similar results were obtained in all repetitions of the same conditions, so that it is only necessary to give two sets of observations here by way of illustration.

In the first set the conditions were those of dry contact throughout the pile of specimens, and therefore surface, as well as body, reflection effects play their part.

Table I.

Specimen added.	t = total thickness.	D observed.	D calculated.	Obs. − calc. Δ.
c	0·989	0·278	0·280	−0·002
e	1·941	0·480(5)	0·480	+0·0005
d	2·927	0·669	0·667	+0·002
a	3·889	0·836	0·837	−0·001
b	4·876	1·006	1·003	+0·003
e only	0·952	0·272	0·272	0·000

Plotting logs D against logs t, the values all fall well on a straight line, represented by the equation

$$\log \mathrm{D} = 0{\cdot}80 \log t + \bar{1}{\cdot}451\,;$$

hence

$$\mathrm{D} = 0{\cdot}282\,(5)\, t^{0{\cdot}80}.$$

The values in the fourth column are calculated from this equation, and the (last) column of differences shows how accurately it represents the observations. These results and several other similar sets of readings show that the empirical rule discovered by Bloch and Renwick can be relied upon well within the limits of experimental error in this case, and we feel, therefore, justified in using it as an interpolation formula for calculating the relative obstructance logarithms of any desired intermediate thicknesses.

From the above equation we now calculate the relative obstructance values for one and for two specimens of the same thickness as the 5 cm. diameter instrument opal (viz., 0·969 mm.).

The results are:—

One specimen, 0·969 mm. thick, Rel. D $= 0{\cdot}275\,(4)$; $\mathrm{O}_1 = 1{\cdot}886$.
Two specimens, 0·969 mm. thick, Rel. D $= 0{\cdot}479\,(5)$; $\mathrm{O}_2 = 3{\cdot}016$.

Assuming that the instrument opal does completely diffuse the light reaching the specimens, we next calculate the rejectance of the diffuser towards the specimens for dry contact from the equation

$$\mathrm{O}_1{}^2 - \mathrm{O}_2 = \rho^2, \text{ i.e. } 1{\cdot}886^2 - 3{\cdot}016.$$

This gives the value of $\rho = 0{\cdot}734$.

From the same empirical formula we calculated the values in Table II for a set of specimens all exactly 1 mm. thick, in contact with the photometer opal (0·969 mm. thick). A trifling error is introduced in making these calculations when dealing with dry contact, by reason of the fact that part of the rejectance and obstructance is due to the air-opal surfaces, and this part of the whole will depend on the number of surfaces, not on the thickness of the specimens. Owing, however, to the fact that our measured specimens were all approximately equal to 1 mm. in thickness, the total correction for thickness is very small, and therefore the neglect of this trifling error is quite justifiable here. The effects of air-surface reflections among a pile of dry specimens become noticeable, however, if the specimens vary considerably in thickness. We then find that the series of observations do not fall well on a straight line when logs D are plotted against logs t, whereas they do so if these reflections are eliminated by uniting the specimens with films of cedar-wood oil (see the second set of observations recorded later).

Table II.

Thickness.	Log D.	Log O_t.	O_t.
mm.			
1	$\bar{1}\cdot451$	0·282 (5)	1·916
2	$\bar{1}\cdot6918$	0·492	3·105
3	$\bar{1}\cdot8327$	0·680	
4	$\bar{1}\cdot9327$	0·856 (5)	
5	0·0102	1·023	
6	0·0736	1·185	
7	0·1271	1·340	

Using the values of O given in Table II for 1 and 2 mm. specimens, we deduce the value of R_1 for a 1 mm. thick specimen from equation (18)

$$R_1^2(O_1^2-\rho^2)-R_1(O_2-1)\rho-(O_1^2-O_2) = 0,$$

taking $\rho = 0{\cdot}734$; the result is $R_1 = 0{\cdot}738$.

From $K = (O_2+1)/O_1 = 2\cosh\lambda$ we have $K = 2{\cdot}142$, $\cosh\lambda = 1{\cdot}071$, $\sinh\lambda = 0{\cdot}3838$, $e^{-\lambda} = 0{\cdot}6875$, and from equation (15), $R_\infty = 0{\cdot}883$.

We are now in a position to test our complete theory by calculating values for the remainder of the specimens included in Table II from formula (12) for O_t.

The formula is

$$O_t = [O_1\sinh\lambda t-\sinh\lambda(t-1)]/\sinh\lambda.$$

The next table (Table III) is calculated from these data.

Table III.

t.	Log $\sinh\lambda t$.	$O_1\sinh\lambda t$.	$\sinh\lambda(t-1)$.	Log $O_t = D_t$.
3	0·1392	2·640	0·822	0·675
4	0·3282	4·081	1·378	0·848
5	0·5029	6·101	2·129	1·015
6	0·6713	8·991	3·184	1·180
7	0·8367	13·16	4·691	1·344

If the values of log O_t in this third Table are compared with those in Table II (which are based directly upon observed values) it will be seen that the agreement is remarkably good; in no case is the difference so great as 0·01.

The agreement between theory and observation was further checked by plotting the logs of D_t taken from Table III against logs t, including also the assumed values for 1 and 2 mm. specimens. The equation to the line passing best through the plotted values was found to be identical with that found for the actual observations, viz. :—

$$\log D = 0{\cdot}80 \log t + \bar{1}{\cdot}451.$$

The next illustration is one of a number of sets of observations made upon the same specimens with all surfaces in optical contact by means of films of cedar-wood oil, so as practically to eliminate the air-opal surface reflections.

Table IV contains a summary of the observed values, and a comparison of these with those calculated from the empirical formula

$$\log D = 0{\cdot}86 \log t + \bar{1}{\cdot}431.$$

Table IV.

Added specimen.	Total thickness. mm.	Log O_t = D observed.	Calc. D.	D obs. − D calc. Δ.
f	0·655	0·185	0·187 (5)	−0·0025
e alone	0·952	0·252	0·258 (5)	−0·0065
c alone	0·989	0·270	0·267	+0·003
c + *e*	1·941	0·477	0·477	0·000
d	2·927	0·683	0·679 (5)	+0·003 (5)
a	3·889	0·869	0·867 (5)	+0·001 (5)
b	4·876	1·047	1·054	−0·007
f	5·531	1·174	1·174	0·000

Once again the agreement between the observations and the empirical rule is extremely good.

The following values of O were calculated as before for one and two specimens, 0·969 mm. thick :—

$$O_{0.969} = 1{\cdot}830 \, ; \qquad O_{1.938} = 2{\cdot}997,$$

whence (for oil contact) $\rho^2 = 1{\cdot}830^2 - 2{\cdot}997$; $\rho = 0{\cdot}594$.

The next Table, No. V, gives the values for thicknesses, 1 up to 7 mm., calculated from the same empirical equation :—

$$\log D = 0{\cdot}86 \log t + \bar{1}{\cdot}431.$$

Table V.

t.	Log D.	D.	O.
mm.			
1	$\bar{1}\cdot431$	0·270	1·861
2	$\bar{1}\cdot6899$	0·490	3·088
3	$\bar{1}\cdot8413$	0·694	
4	$\bar{1}\cdot9488$	0·889	
5	0·0321	1·076	
6	0·1002	1·260	
7	0·1578	1·438	

As in the first example, we evaluate from the known values of ρ for oil contact, and of O for 1 and 2 mm. thickness, the constants of the theoretical equation for O_t thus :—

$$R_1^2(1{\cdot}861^2-0{\cdot}594^2)-R_1(3{\cdot}088-1)\,0{\cdot}594-(1{\cdot}861^2-3{\cdot}088)=0,$$

$$R_1 = 0{\cdot}5995\,;$$

$$K = 2\cosh\lambda = (O_2+1)/O_1 = 2{\cdot}196\,; \qquad e^{-\lambda} = 0{\cdot}6445,$$

$$\sinh\lambda = 0{\cdot}4534, \qquad \lambda = 0{\cdot}4392.$$

Also, from equation (15),

$$R_\infty = O_1R_1/[O_1-(1-\rho R_1)\,e^{-\lambda}] = 0{\cdot}7716.$$

which leads to the following values for Schuster's constants :—

$$\alpha = 0{\cdot}1289\,; \qquad s = 3{\cdot}350\,; \qquad k = 0{\cdot}05662.$$

The difference between R_1 for oil and air contact shows that about 14 per cent. is rejected by the first air surface, and therefore the maximum total rejectance of a thick solid block of this opal = 0·91.

The next Table (No. VI) is calculated from the above data on the formula

$$O_t = [O_1\sinh\lambda t - \sinh\lambda\,(t-1)]/\sinh\lambda.$$

Table VI.

t.	Log sinh λt.	O_1 sinh λt.	Sinh λ $(t-1)$.	Log $O_t = D_t$.
3	0·2386	3·223	0·995	0·692
4	0·4484	5·225	1·732	0·887
5	0·6468	8·250	2·808	1·080
6	0·8407	12·890	4·434	1·271
7	1·0326	20·05	6·929	1·462

Comparing these values of log O_t with the values in Table V (based on the experimental observations), it is seen that, excluding the 7 mm. thickness, which is beyond the range of our measurements, the agreement is remarkably close throughout; and also, how good an approximation the empirical rule is to the full theoretically deduced equation is strongly brought out.

These and other similar sets of observations of the most careful kind appear to us amply to demonstrate the soundness of our theoretical conclusions and to confirm Bloch and Renwick's observations on the practical utility of the empirical formula if used within certain limits. We have endeavoured to discover how closely the theoretical laws as here set out can be approximated to by use of the empirical rule, by comparing the results for a number of imaginary cases. It is clear that any of our formulæ for Ω_t, C_t and O_t become identical with the well-known ordinary law of absorption in non-scattering media if Ω_1, C_1 and O_1 respectively $= e^{\lambda}$.

But if R_1 is not zero and $O_1 = e^{\lambda}$, $(1-\rho R_1)\Omega_1 = e^{\lambda}$ and consequently $\rho = (\Omega_1 - e^{\lambda})/\Omega_1 R_1$, which is only another expression for R_{∞}.

It is clear, therefore, that if by chance the rejectance of the photometer diffuser is equal to the maximum possible rejectance of the material under measurement, the results will follow the ordinary law of absorption for non-scattering media and the empirical rule $D = at^b$ will be exactly followed with $b = 1$.

Although these precise conditions are not often realised, practical results and a study of imaginary cases indicate that the empirical rule holds very well for five and sometimes more units of thickness when a photometer diffuser of high rejectance such as white solid opal glass about 1 or more millimetres thick is used, whereas it fails to hold if R_1 is much higher than ρ.

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